Om and Statefinder Diagnostics of Anisotropic $f(R, \mathcal{L}_m)$ Gravity

Prof. Dr. Dnyaneshwar D. Pawar
Professor and Director
School of Mathematical Sceiences,
Swami Ramanand Teerth Marathwada University, Nanded, MH-India

CosmoFondue, 12, Jun 2025

$f(R, \mathcal{L}_m)$ Theory of Gravity

- The action for $f(R, \mathcal{L}_m)$ gravity, proposed by Harko and Lobo, is:

$$S = \int f(R, \mathcal{L}_m)\sqrt{-g}\, d^4x, \tag{1}$$

where f is a general function of the Ricci scalar R and the matter Lagrangian $\mathcal{L}_m$.

- The Ricci scalar is defined as:

$$R = g^{ij} R_{ij}. \tag{2}$$

- The Ricci tensor is given by:

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ki} + \Gamma^\lambda_{ij}\Gamma^k_{\lambda k} - \Gamma^k_{j\lambda}\Gamma^\lambda_{ki}. \tag{3}$$

$f(R, \mathcal{L}_m)$ Theory of Gravity

- Varying the action with respect to the metric yields the field equations:

$$f_R R_{ij} - \frac{1}{2}(f - f_{\mathcal{L}_m}\mathcal{L}_m)g_{ij} + (g_{ij}\Box - \nabla_i\nabla_j)f_R = \frac{1}{2}f_{\mathcal{L}_m}T_{ij}. \tag{4}$$

- Here, the partial derivatives of f are:

$$f_R = \frac{\partial f(R, \mathcal{L}_m)}{\partial R},$$
$$f_{\mathcal{L}_m} = \frac{\partial f(R, \mathcal{L}_m)}{\partial \mathcal{L}_m},$$
$$\Box = \nabla_i\nabla^i.$$

- The energy-momentum tensor is defined by:

$$T_{ij} = g_{ij}\mathcal{L}_m - 2\frac{\partial \mathcal{L}_m}{\partial g^{ij}}. \tag{5}$$

$f(R, \mathcal{L}_m)$ Theory of Gravity

- The covariant divergence of the energy-momentum tensor is given by:

$$\nabla^i T_{ij} = 2\nabla^i \ln(f_{\mathcal{L}_m}) \frac{\partial \mathcal{L}_m}{\partial g^{ij}}. \tag{6}$$

- Contracting the field equations leads to the trace relation:

$$R f_R + 3\Box f_R - (f - f_{\mathcal{L}_m} \mathcal{L}_m) = \frac{1}{2} f_{\mathcal{L}_m} T, \tag{7}$$

where $T = g^{ij} T_{ij}$ is the trace of the energy-momentum tensor.

- The d'Alembertian operator $\Box$ acting on a scalar F is defined as:

$$\Box F = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j F).$$

Equation of Motion in $f(R, \mathcal{L}_m)$ Gravity

- Anisotropic cosmological models introduce extra degrees of freedom and richer dynamics compared to isotropic ones like FLRW, though they are more challenging to analyze.
- We consider a homogeneous and anisotropic LRS Bianchi Type-I spacetime:

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)(dy^2 + dz^2), \tag{8}$$

where $A(t)$ and $B(t)$ are scale factors in cosmic time.

- For $A(t) = B(t)$, this reduces to the standard FLRW metric.
- The Ricci scalar for this metric is:

$$R = -2\left[\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B} + 2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2}\right]. \tag{9}$$

- Assuming a perfect fluid distribution, the energy-momentum tensor is:

$$T_{ij} = (p + \rho)u_i u_j + p g_{ij}, \tag{10}$$

where ρ is energy density, p is pressure, and $u^i = (1, 0, 0, 0)$ is the comoving four-velocity.

Equation of Motion in $f(R, \mathcal{L}_m)$ Gravity

- The modified Friedmann-like field equations in $f(R, \mathcal{L}_m)$ gravity for the Bianchi Type-I metric are given as:

$$-\left(\frac{\ddot{A}}{A} + 2\frac{\dot{A}\dot{B}}{AB}\right) f_R - \frac{1}{2}(f - f_{\mathcal{L}_m}\mathcal{L}_m) - 2\frac{\dot{B}}{B}\dot{f}_R - \ddot{f}_R = \frac{1}{2}f_{\mathcal{L}_m}p, \tag{11}$$

$$-\left(\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{\dot{A}\dot{B}}{AB}\right) f_R - \frac{1}{2}(f - f_{\mathcal{L}_m}\mathcal{L}_m) - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{f}_R - \ddot{f}_R = \frac{1}{2}f_{\mathcal{L}_m}p, \tag{12}$$

$$-\left(\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B}\right) f_R - \frac{1}{2}(f - f_{\mathcal{L}_m}\mathcal{L}_m) - \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\dot{f}_R = \frac{1}{2}f_{\mathcal{L}_m}\rho. \tag{13}$$

- Here, $f_R = \partial f/\partial R$, $f_{\mathcal{L}_m} = \partial f/\partial \mathcal{L}_m$, and dots denote derivatives with respect to cosmic time t.

Cosmological Solutions for $f(R, \mathcal{L}_m)$ Gravity

- We consider a specific form of the function:

$$f(R, \mathcal{L}_m) = \frac{R}{2} + \mathcal{L}_m^{\xi} + \zeta, \tag{14}$$

where ξ and ζ are constants. GR is recovered when $\xi = 1$ and $\zeta = 0$.

- Assuming a dust-like Universe ($\mathcal{L}_m = \rho$), the modified field equations become:

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \zeta - (1-\xi)\rho^{\xi} = \xi\rho^{\xi-1}p, \tag{15}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \zeta - (1-\xi)\rho^{\xi} = \xi\rho^{\xi-1}p, \tag{16}$$

$$\frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} - \zeta = (1-2\xi)\rho^{\xi}. \tag{17}$$

Cosmological Solutions for $f(R, \mathcal{L}_m)$ Gravity

- To solve the system, we use a deceleration parameter approach as in Tiwari et al.**[16]**, where:

$$q = \frac{-a\ddot{a}}{\dot{a}^2}, \quad \text{with} \quad q(t) = \alpha - \frac{\beta}{H}. \tag{18}$$

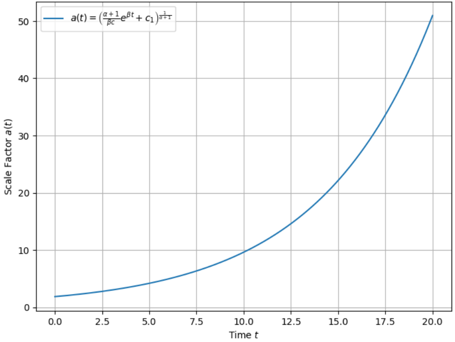
- The average scale factor is:

$$a = (AB^2)^{1/3}, \tag{19}$$

which solves to:

$$a = \left(\frac{\alpha + 1}{\beta c} e^{\beta t} + c_1 \right)^{\frac{1}{\alpha+1}}. \tag{20}$$

Cosmological Solutions for $f(R, \mathcal{L}_m)$ Gravity



figureEvolution of the scale factor $a(t)$ with cosmic time t.

- The evolution of the scale factor $a(t)$ provides insight into the Universe's expansion dynamics.
- Initially, the Universe exhibits slower expansion, corresponding to a decelerated phase dominated by matter.
- As time progresses, the scale factor increases rapidly, signaling a transition to an accelerated expansion phase.
- This late-time acceleration aligns with current cosmological observations of dark energy-driven expansion.

Cosmological Solutions for $f(R, \mathcal{L}_m)$ Gravity

- Using the power-law relation, the metric functions become:

$$B(t) = \left(\frac{\alpha + 1}{\beta c} e^{\beta t} + c_1 \right)^{\frac{3}{(\alpha+1)(2+n)}}, \tag{21}$$

$$A(t) = \left(\frac{\alpha + 1}{\beta c} e^{\beta t} + c_1 \right)^{\frac{3n}{(\alpha+1)(2+n)}}. \tag{22}$$

- The line element now simplifies to:

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)(dy^2 + dz^2), \tag{23}$$

showing anisotropic expansion with time-dependent scale factors.

Hubble Parameter

- Redshift z is related to the scale factor via:

$$a = \frac{1}{1+z}, \quad \text{with } a = 1 \text{ at } z = 0.$$

- For the model $f(R, \mathcal{L}_m) = \frac{R}{2} + \mathcal{L}_m^{\xi} + \zeta$, the Hubble parameter becomes:

$$H(z) = \frac{\beta}{\alpha+1}\left[1 - c_1(1+z)^{\alpha+1}\right],$$

$$H_0 = \frac{\beta}{\alpha+1}(1 - c_1).$$

- Model predictions are compared with 57 observational $H(z)$ data points from DA and BAO methods.

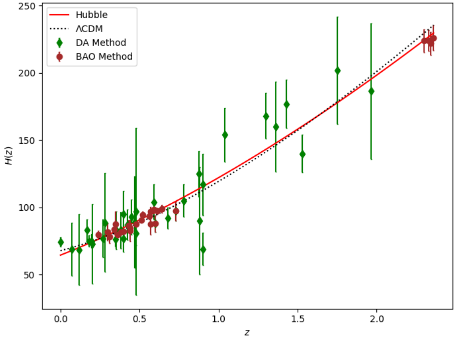


figure: Best-fit curve of $H(z)$ vs redshift. Red: model; green diamonds: DA; red circles: BAO.

- Best-fit values:
  - $\alpha = 0.542^{+0.019}_{-0.022}$
  - $\beta = 52.9^{+2.3}_{-2.7}$
  - $c_1 = -0.877^{+0.055}_{-0.058}$
  - $H_0 = 64.39^{+0.04}_{-0.47}$ km/s/Mpc
- Model accuracy:
  - $R^2 = 0.9321$
  - RMSE = 11.0716
- ΛCDM comparison:

$$H(z) = H_0\sqrt{\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0}}$$

where $H_0 = 67.8$, $\Omega_{m0} = 0.3$, $\Omega_{\Lambda 0} = 0.7$.

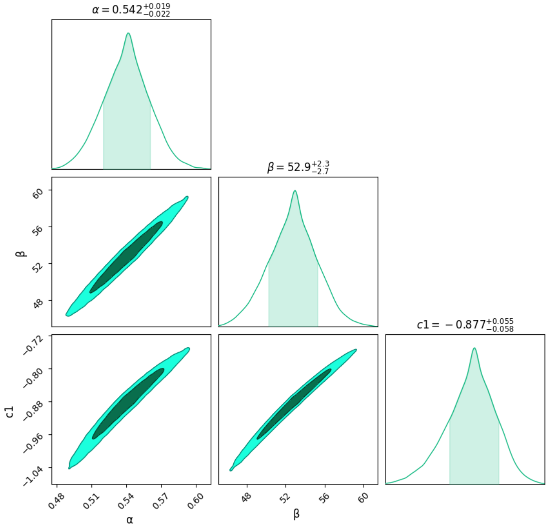


Figure: Confidence contours for α, β, and c_1 at 1σ and 2σ

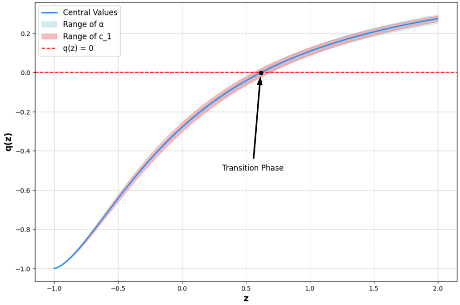
Deceleration Parameter in Redshift

- From the scale factor–redshift relation, the deceleration parameter is expressed as:

$$q(z) = \alpha - \frac{\alpha + 1}{1 - c_1(1+z)^{\alpha+1}}$$

- This function describes the evolution of the expansion rate — identifying transitions between deceleration and acceleration.
- The parameters used are:

$$\alpha = 0.542, \quad c_1 = -0.877, \quad H_0 = 64.39\,\text{km/s/Mpc}$$



figurePlot of $q(z)$ vs redshift z.

Deceleration Parameter in Redshift

- The plot of $q(z)$ shows that at low redshift ($z \approx -1$), $q \approx -1$, indicating an accelerating phase.
- As z increases, $q(z)$ transitions to positive values, indicating a decelerating phase in the early universe.
- The red dashed line at $q = 0$ marks the boundary between acceleration and deceleration.
- The black dot marks the transition point around $z \approx 0.5$.
- Shaded regions:
  - Blue: uncertainty due to α
  - Red: uncertainty due to c_1
- The model yields:

$$R^2 = 0.9321, \quad \text{RMSE} = 11.0716$$

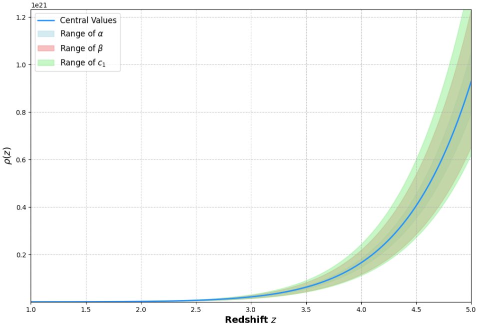
showing strong agreement with observational data and robustness of parameter constraints.

Energy Density as a Function of Redshift

- The energy density $\rho(z)$ is derived as:

$$\rho(z) = \left\{ \frac{9\beta^2 \left[(1+z)^{-(\alpha+1)} - c_1 \right]^2}{(2+n)^2 (1-2\xi)(\alpha+1)^2} (1+2n)(1+z) \right.$$

- This describes how energy density evolves with redshift based on model parameters.



figureEnergy density $\rho(z)$ vs redshift.

Energy Density as a Function of Redshift

- The plot of energy density illustrates how $\rho(z)$ increases rapidly as redshift increases, consistent with a denser early universe.
- The blue curve shows the best-fit density evolution; shaded regions indicate uncertainty:
  - Blue shaded: uncertainty from α
  - Red shaded: uncertainty from β
  - Green shaded: uncertainty from c_1
- The growth in $\rho(z)$ is highly sensitive to c_1, especially at high z, which corresponds to early universe epochs.
- This behavior supports the standard picture of structure formation, where the universe was more compact and energetic at earlier times.
- The expression fits well with observational trends and reflects early-universe matter dominance.

Pressure as a Function of Redshift

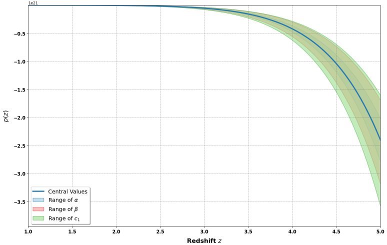
- The pressure $p(z)$ is given by the expression:

$$p(z) = \frac{1}{\xi}\left\{\frac{9\beta^2\left[(1+z)^{-(\alpha+1)}-c_1\right]^2}{(2+n)^2(1-2\xi)(\alpha+1)^2}(1+2n)(1+z)^{2(\alpha+1)}-\frac{\zeta}{1-2\xi}\right\}^{\frac{1-\xi}{\xi}}$$

$$\times\left\{\frac{6\beta^2\left[(1+z)^{-(\alpha+1)}-c_1\right]}{(2+n)(\alpha+1)}(1+z)^{\alpha+1}\right.$$

$$+\left[9-2(\alpha+1)(2+n)\right]\frac{3\beta^2\left[(1+z)^{-(\alpha+1)}-c_1\right]^2}{(2+n)^2(\alpha+1)^2}(1+z)^{2(\alpha+1)}$$

$$\left.-\zeta-(1-\xi)\left[\frac{9\beta^2\left[(1+z)^{-(\alpha+1)}-c_1\right]^2}{(2+n)^2(1-2\xi)(\alpha+1)^2}(1+2n)(1+z)^{2(\alpha+1)}-\frac{\zeta}{1-2\xi}\right]\right\}$$



Pressure $p(z)$ vs redshift.

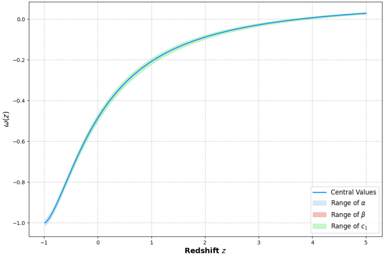
Pressure as a Function of Redshift

- The above Figure shows the behavior of pressure $p(z)$ across redshift z.
- At higher redshifts, pressure becomes more negative — indicating dark energy dominance.
- The green-shaded region dominates at high z, highlighting the strong sensitivity of pressure to c_1 during early cosmic times.
- This behavior aligns well with the theoretical expectation of negative pressure driving accelerated expansion in the late universe.

Equation of State Parameter $\omega(z)$

- The equation of state (EoS) parameter $\omega(z)$ is:

$$\omega(z) = -1 + \xi + \frac{1}{\xi}\left\{\frac{9\beta^2\left[(1+z)^{-(\alpha+1)} - c_1\right]^2}{(2+n)^2(1-2\xi)(\alpha+1)^2}(1+2n)(1+z)^{2(\alpha+1)} - \frac{\zeta}{1-2\xi}\right\}^{-1}$$
$$\times\left\{\frac{6\beta^2\left[(1+z)^{-(\alpha+1)} - c_1\right]}{(2+n)(\alpha+1)}(1+z)^{\alpha+1} + [9 - 2(\alpha+1)(2+n)]\right.$$
$$\left.\times\frac{3\beta^2\left[(1+z)^{-(\alpha+1)} - c_1\right]^2}{(2+n)^2(\alpha+1)^2}(1+z)^{2(\alpha+1)} - \zeta\right\}$$



parameter $\omega(z)$ vs redshift.

Equation of State Parameter $\omega(z)$

- Figure 19 illustrates the evolution of the EoS parameter $\omega(z)$ with redshift z.
- The shaded regions indicate uncertainties due to:
    - Light blue: variation in α
    - Pink: variation in β
    - Green: variation in c_1
- At low redshift ($z \approx 0$), $\omega \approx -1$, consistent with dark energy behavior.
- As z increases, $\omega(z)$ rises gradually, indicating a transition toward matter-dominated conditions.
- The narrow shaded bands reflect stable model predictions with well-constrained parameters.

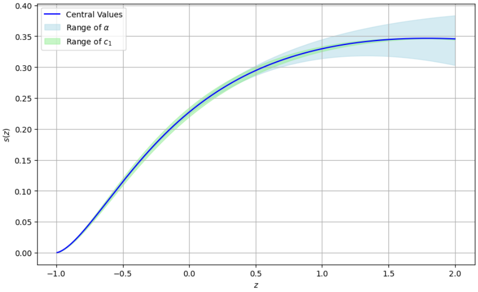
Statefinder Diagnostic : The Parameter $s(z)$

- The statefinder parameters are defined as:

$$r = \frac{\dddot{a}}{aH^3}, \quad s = \frac{r-1}{3\left(q - \frac{1}{2}\right)}$$

- The expression for $s(z)$ is:

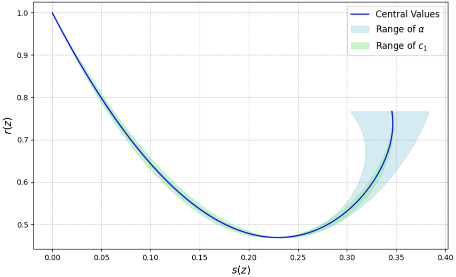
$$s(z) = \frac{2}{-3\left(3 + (2\alpha - 1)c_1(1+z)^{\alpha+1}\right)\left(1 - c_1(1+z)^{\alpha+1}\right)^2} \times \Bigg\{ (1+z)^{\alpha+1}(\alpha+1)^2\left((1+z)^{-(\alpha+1)} - c_1\right) - 3\alpha(\alpha+1)\left((1+z)^{-(\alpha+1)} - c_1\right)^2(1+z)^{2\alpha+2} + \alpha(2\alpha+1)\left((1+z)^{-(\alpha+1)} - c_1\right)^3(1+z)^{3\alpha+3} - \left(1 - c_1(1+z)^{\alpha+1}\right)^3 \Bigg\}$$



figurePlot of $s(z)$ vs redshift z.

Comparison of r and s

- The statefinder pair (r, s) helps distinguish between various dark energy models.
- The point $(r, s) = (1, 0)$ corresponds to the standard ΛCDM model.
- The region where $r < 1$ and $s > 0$ corresponds to the Quintessence model.



figurePlot of $r(z)$ vs $s(z)$

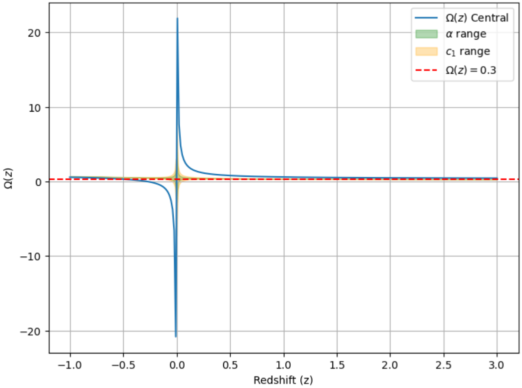
Om Diagnostics

- The Om diagnostic helps distinguish dark energy models via expansion rate.
- Defined as:

$$\Omega(z) = \frac{\left(\frac{H(z)}{H_0}\right)^2 - 1}{(1+z)^3 - 1}$$

- Calculated Om diagnostics:

$$\Omega(z) = \frac{2c_1\left[1-(1+z)^{\alpha+1}\right] + c_1^2\left[(1+z)^{\alpha+1}\right.}{(1-c_1)^2\left((1+z)^3-1\right)}$$



figurePlot of $\Omega(z)$ vs redshift z.

Discussion and Conclusion

- A LRS Bianchi Type-I cosmological model with perfect fluid was investigated in the $f(R, \mathcal{L}_m)$ gravity framework.
- Exact field equations were derived and cosmological parameters like energy density, pressure, equation of state (EoS), Hubble parameter, deceleration parameter, expansion scalar, and shear scalar were evaluated.
- The Hubble parameter $H(z)$ was fitted to 57 observational data points with an $R^2 = 0.9321$, showing strong agreement with the OHD data.
- Best-fit parameters:
  - $\alpha = 0.542^{+0.019}_{-0.022}$, $\beta = 52.9^{+2.3}_{-2.7}$
  - $c_1 = -0.877^{+0.055}_{-0.058}$, $H_0 = 64.39^{+0.04}_{-0.47}$ km/s/Mpc
- The model's predictions are consistent with the standard ΛCDM model, particularly at low redshifts.
- The deceleration parameter $q(z)$ shows a transition from deceleration to acceleration near $z \approx 0.5$, supported by confidence regions in α and c_1.

Discussion and Conclusion

- Evolution of physical parameters:
  - $\rho(z)$: Increases with redshift – denser early universe.
  - $p(z)$: Transitions to negative values – consistent with dark energy.
  - $\omega(z)$: Evolves from -1 (dark energy) toward 0 – indicating Quintessence phase.
- Statefinder diagnostic:
  - The pair $(r, s) = (1, 0)$ confirms ΛCDM consistency.
  - For best-fit values, model indicates Quintessence-like behavior for $r < 1$, $s > 0$.
- Om diagnostic:
  - $\Omega(z)$ closely matches ΛCDM at higher z.
  - Deviations near $z = 0$ suggest rapid late-time expansion.
- Overall, the model agrees with current cosmological observations and provides a viable alternative to ΛCDM within modified gravity frameworks.

Thank You!