Time-reversed Stochastic Inflation

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Starobinsky inflation

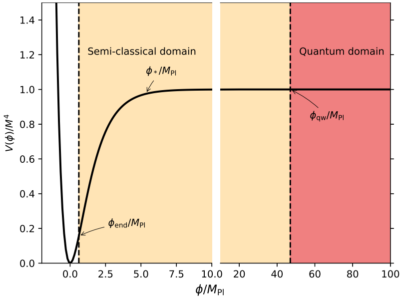
- Prototypical of the favoured models: $V(\phi) = M^4 \left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_{\rm Pl}}}\right)^2$

Classical slow-roll evolution

- FLRW + Klein-Gordon equation

$$H^2 = \frac{1}{3}\left(\dot{\phi}^2/2 + V(\phi)\right)$$
$$\frac{\ddot{a}}{a} = -\frac{1}{3}\left(\dot{\phi}^2 - V(\phi)\right)$$
$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

- Time measured in numbers of e-folds $N \equiv \ln a$
- $H = \dot{a}/a \simeq \text{cste}$ hence quasi de Sitter evolution
- Quantization of linear perturbations $\delta\phi \longrightarrow$ cosmological predictions



Starobinsky inflation

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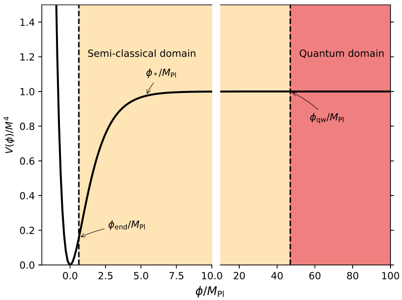
Quantum regime

- Quantum fluctuations dominate

$$\delta\phi_{\rm cl} < \delta\phi_{\rm qu} \simeq \frac{H(\phi)}{2\pi}$$

- Large primordial curvature perturbations can be generated
- Primordial Black Holes, scalar induced GWs, etc
- Ultra-slow-roll (USR) phases
- Need for a non-perturbative approach $\longrightarrow$ stochastic inflation formalism



If our universe were born at $\phi > \phi_{\rm qw}$?

- Semi-classical evolution from $\phi_{\rm qw}$ to $\phi_* \sim 10^{17}$ e-folds
- Quantum domain may alter structures only on distances $d > 10^{10^{16}}\,{\rm Gpc}$

Eternal inflation

Why bothering? [Ringeval:2010, Ringeval:2019, Blachier:2023]

- Access to largest length scales of our universe
- Super-Hubble fluctuations may create $\Omega_{\rm K}, \Omega_\Lambda$

Eternal inflation and multiverse structure

- Large quantum fluctuations counteract classical drift
- Inflation is eternal: some regions of the Universe always inflate i.e. never reach $\phi_{\rm qw}$

Can Stochastic Inflation still provide some insights?

Stochastic Inflation formalism [Starobinsky:1986, Goncharov:1987, Linde:1993]

An effective field theory in curved space-time

- Separation between large and small length scales

$$\hat{\phi} = \hat{\phi}_{\rm cg} + \hat{\phi}_{\rm uv} = \hat{\phi}_{\rm cg} + \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} W\left(\frac{k}{k_\sigma}\right)\left[\hat{a}_{\boldsymbol{k}}\varphi_k(N)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{k}}\varphi^*_k(N)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right]$$

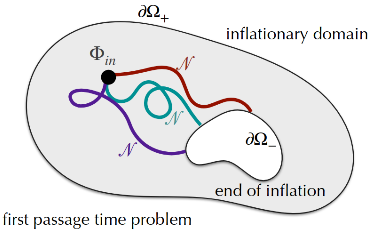
- Dynamics of the coarse-grained field

$$\frac{\partial\hat{\phi}_{\rm cg}}{\partial N} = \hat{\Gamma}_{\rm cg} + \hat{\xi}_\varphi \qquad (1)$$

- In quasi-de Sitter spacetime, quantization over Bunch-Davies vacuum

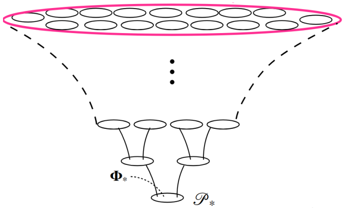
- At $k_\sigma \ll aH$, quantum operator $\longrightarrow$ Gaussian noise [Lesgourgues:1996, Grain:2017]
- Langevin equation for the coarse-grained field

$$\boxed{\frac{\mathrm{d}\phi_{\rm cg}}{\mathrm{d}N} = -\frac{V'(\phi_{\rm cg})}{3H^2} + \frac{H}{2\pi}\xi(N)}$$



Separate universe picture

- $\delta\phi_{\mathrm{cg}}$ create inhomogeneities in regions of size k_σ^{-1}
- Spacetime as an ensemble of independent, locally homogeneous and isotropic Hubble sized patches

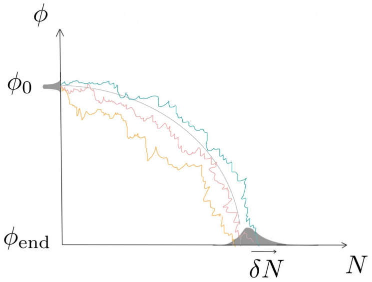


Stochastic-δN formalism [Vennin:2015, Pattison:2017]

- Curvature perturbation ζ is the local amount of expansion

$$\zeta_{\mathrm{cg}} = N - \langle N \rangle$$

- Non-perturbative approach



Stochastic inflation in the semi-infinite flat potential

- For $\phi > \phi_{\text{qw}}$, pure Brownian motion with diffusion $G \equiv H/(2\pi)$
- Forward transition probability from Fokker-Planck equation with absorbing boundary at ϕ_{qw}

$$P(\phi, N|\phi_0, N_0) = \frac{1}{\sqrt{2\pi}G\sqrt{N-N_0}}\left[e^{-\frac{(\phi-\phi_0)^2}{2G^2(N-N_0)}} - e^{-\frac{(\phi-2\phi_{\text{qw}}+\phi_0)^2}{2G^2(N-N_0)}}\right]$$

- How the N_{qw} are distributed?

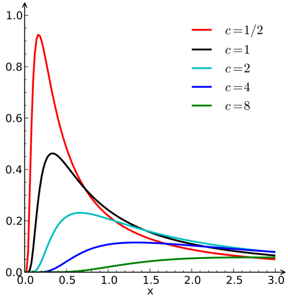
$$P_{\text{LT}}(N_{\text{qw}}|\phi_0) = \frac{\phi_0-\phi_{\text{qw}}}{\sqrt{2\pi}GN_{\text{qw}}^{3/2}}e^{-\frac{(\phi_0-\phi_{\text{qw}})^2}{2G^2N_{\text{qw}}}}$$

Eternal inflation: $\langle N_{\text{qw}}\rangle = \infty$

- Other tilted and/or bounded potentials are fine
- Yet, generic feature of inflation (and plateau-type models)

How to determine ζ and its probability distribution?

- Observers are attached to N_{qw}
- Let's reverse time: starting from N_{qw} and evolving back to N_0



Time reversal of stochastic inflation

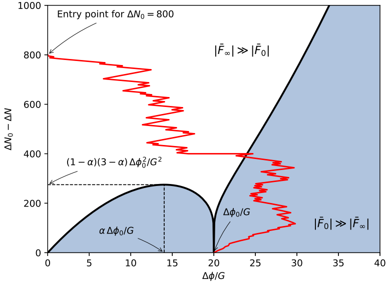
Time-reversal of diffusion: $\bar{P}(y, s|x, t, x_0, t_0)$ with $t > s > t_0$ [Schrödinger:1933, Nagasawa:1964, Chung:1969, Anderson1982]

- Modified Fokker-Planck equation

$$-\frac{\partial}{\partial s}\bar{P}(y, s|x, t, x_0, t_0) = \frac{\partial}{\partial y}\left[F(y, s) - G^2\frac{\partial}{\partial y}\ln P(y, s|x_0, t_0)\right]\bar{P}(y, s|x, t, x_0, t_0) + \frac{G^2}{2}\frac{\partial^2}{\partial y^2}\bar{P}(y, s|x, t, x_0, t_0)$$

With respect to N_{qw}

- Reverse e-folds time: $\Delta N \equiv N_{\mathrm{qw}} - N$
- Field values in reference to the quantum wall: $\Delta\phi \equiv \phi - \phi_{\mathrm{qw}}$
- In the forward picture, time-reversal may be interpreted as a conditioning and partitioning by the lifetimes: $\Delta N_0 = N_{\mathrm{qw}} - N_0$



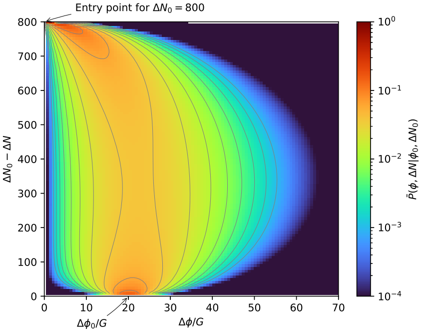
Reverse probability distribution

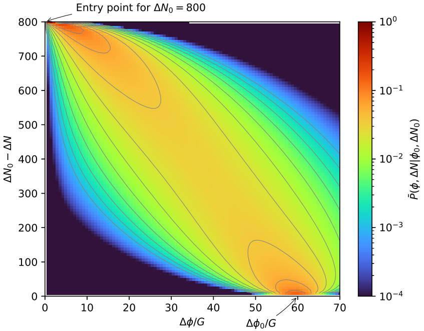
Exact probability distribution *via* Girsanov's theorem [Mazzolo:2024]

$$\bar{P}(\Delta\phi, \Delta N|\Delta\phi_0, \Delta N_0) = \frac{\sqrt{2/\pi}}{\tau^{3/2}\sqrt{1-\tau}} \frac{\chi}{\chi_0 G\sqrt{\Delta N_0}} \sinh\left[\frac{\chi\chi_0}{(1-\tau)\Delta N_0}\right] e^{-\frac{\chi^2+\tau^2\chi_0^2}{2\tau(1-\tau)\Delta N_0}}$$

with $\tau \equiv \Delta N/\Delta N_0$ and $\chi \equiv \Delta\phi/G$.

- $\Delta\phi_0 < G\sqrt{\Delta N_0}$: diffusion dominates
- $\Delta\phi_0 > G\sqrt{\Delta N_0}$: fluxing takes over





Time-reversed stochastic δN formalism

Reverse processes are conditioned by the lifetime ΔN_0

- At given lifetime

$$\langle \Delta N_0 \rangle = \Delta N_0 \quad \text{hence} \quad \zeta|_{\Delta N_0} = \langle \Delta N \rangle - \Delta N$$

- Partitioning and then resumming:

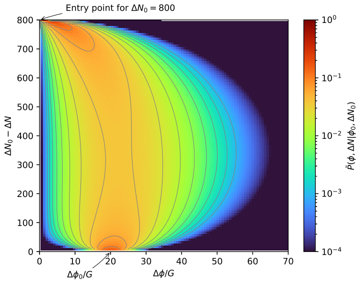
$$\zeta = \bigcup_{\Delta N_0 = 0}^{\infty} \zeta|_{\Delta N_0}$$

To determine the distribution of ζ

- Evaluate $\langle \Delta N \rangle$ at given ϕ, ϕ_0 and ΔN_0
- $\int \mathrm{d}\phi$ to get $P(\zeta|\phi_0, \Delta N_0)$
- Marginalize over lifetimes to get $P(\zeta|\phi_0)$

$$P(\zeta|\phi_0) = \int_0^{+\infty} P(\zeta, \phi_0, \Delta N_0) \boxed{P_{\mathrm{LT}}(\Delta N_0|\phi_0)} \mathrm{d}\Delta N_0$$

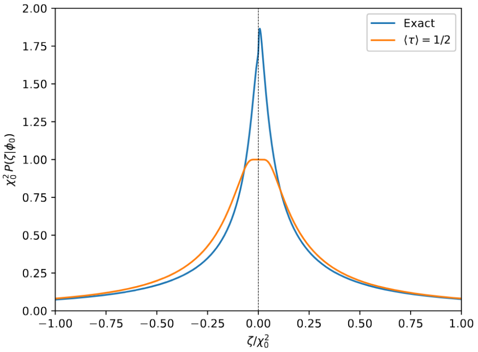
$\Delta N_0 \to \infty$ is included!

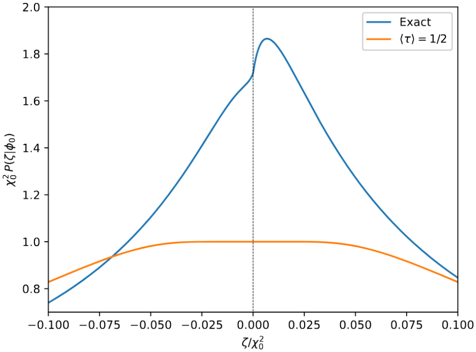


Quantum-generated curvature fluctuations

Normalized, (slightly) skewed distribution towards positive values

- Tails slowly decaying as $1/|\zeta|^{3/2}$
- Maximum at positive value $\zeta_{\rm mode} \simeq 6.9 \times 10^{-3}\chi_0^2$





Conclusion

Reversing time in stochastic inflation

- Mathematically well-defined while being technically more involved
- Automatically enforces the observer point-of-view
- Conditioning by the lifetimes acts as a regulator

Stochastic δN-formalism: $\zeta = \langle \Delta N \rangle - \Delta N$ at given ΔN_0

- Marginalize over lifetimes ΔN_0 in a last step
- $P(\zeta|\phi_0)$ is well-defined even when including $\Delta N_0 \to \infty$
- Heavy tails: $\langle \zeta \rangle = \infty = \langle \zeta^2 \rangle$

In qualitative terms

- $\langle\rangle$ and $\int \mathrm{d}\Delta N_0$ do not commute!
- Nothing bad happens in the flat semi-infinite potential
- Time-reversal as a regularization of eternal inflation