Cosmological Perturbations and Second-Order Effects

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Introduction

Generation of gravitational waves from scalar perturbations at second order in a Friedman Universe.

- Gravitational waves induced by scalar curvature fluctuations are a possible source of the cosmological background of gravitational waves [1].
- While scalar and tensor (gravitational waves) perturbations are decoupled at first order in perturbations theory, gravitational waves are produced by scalar perturbations at second order.
- We try to define a new transverse traceless amplitude which is gauge invariant at second order and can be interpreted as part of the physical gravitational wave.
- This approach, of manipulating the degrees of freedom of the system using gauge invariant quantities, comes from the study of the symplectic structure (Hamiltonian or Poisson), which like in gauge theory in cosmology, is a suitable and consistent approach to reduce the gauge degrees of freedom. [2].

Theoretical Background

In a Friedmann Lemaître Universe, the linearly perturbed metric of pure scalar cosmological perturbations and using conformal time t, is given by:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon a_{(t)} g^{(1)}_{\mu\nu} \tag{1}$$

where the correction term with purely scalar degrees of freedom is:

$$g^{(1)}_{\mu\nu} = \begin{pmatrix} -2\psi & -\partial_i B \\ -\partial_i B & -2\phi\delta_{i,j} + \partial_i\partial_j E - \frac{1}{3}\delta_{i,j}\Delta E \end{pmatrix} \tag{2}$$

and $\Delta = \partial_i\partial^i$, enforcing the traceless component.

On the other hand, we define only first order perturbation in the matter fields, with the energy momentum tensor of **a perfect fluid** given as:

$$\begin{aligned} T_{\mu\nu} &= (\bar{\rho} + \bar{p})(u + \delta u)^{\mu}(\bar{u} + \delta u)^{\nu} + pg^{\mu\nu} \\ &+ (\delta\rho + \delta p)(\bar{u} + \delta u)^{\mu}(\bar{u} + \delta u)^{\nu} + \delta p g^{\mu\nu} \end{aligned} \tag{3}$$

the **anisotropic stress-free** energy momentum tensor, perturbed only to 1st order for a perfect fluid. We have defined $\delta u = (u_0, u_i)$ with $u^i = \partial^i v$ the peculiar velocity, originating from the velocity potential v.

All this leads to the Einstein equations, which up to linear order are:

$$G^{(0)} + G^{(1)} = 8\pi G(T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu})$$

If we expand to second order all tensors, keeping the metric tensor to the given frame, then we have some tensor degrees of freedom (TT) in the Einstein equations originating from the terms quadratic to the scalar fields of the metric and the material fields. These degrees of freedom are not balanced properly.

For all degrees of freedom to be accounted for, we need to introduce the transverse traceless 2nd order mode in the metric as $h^{(2)}_{ij} = h_{ij}$, coming from the equation:

$$G^{(2)}_{(h_{ij})} + G^{(2)}_{(g^{(1)}g^{(1)})} = 8\pi G T^{(2)}_{\mu\nu}$$

which leads to the formation of gravitational waves induced by the quadratic terms of the scalar linear perturbations and matter fields. We can write:

$$G^{(2)}_{(h_{ij})} = 8\pi G T^{(2)}_{\mu\nu} - G^{(2)}_{(g^{(1)}g^{(1)})} := \frac{1}{2} S^{tot}_{\mu\nu}$$

with $S^{tot}_{\mu\nu}$ playing the role of a *source term.*

By extracting the **transverse** and **traceless** (TT) part of these equations, it is known that we are led to the wave equation, satisfied by the tensor perturbations:

$$\frac{1}{2}\Box h_{ij} = \frac{1}{2} {M_{ij}}^{mn} S^{tot}_{mn} \tag{4}$$

with ${M_{ij}}^{mn} = P^m_j P^n_j - \frac{1}{2} P_{ij} P^{mn}$ and

$$P_{ij} = \delta_{ij} - \frac{1}{k^2} k_i k_j$$

the (TT) projector and the wave operator, for an FRW Universe defined as:

$$\Box = \nabla_{\mu}\nabla^{\mu} = \frac{1}{\sqrt{-g}} \partial\mu(\sqrt{-g} g^{\mu\nu}\partial_{\nu})$$

$$\Box = -\frac{1}{a(t)^2}(\partial_t\partial_t + 2\mathcal{H}\partial_t - \Delta)$$

Method and Approach

Since we are interested in the TT component of the source term, we can effectively reduce it, to a term containing only elements which contribute to this component, which is **gauge invariant** by definition.

By algebraically manipulating the source term and by using the **Einstein Equations** at linear order, one can be led to the source term in a form similar to:

$$\begin{aligned} S_{mn} &:= S^N_{mn} - \Box S^{\sigma}_{mn} \\ &= 2(\partial_m\Phi\partial_n\Psi + \partial_n\Phi\partial_m\Psi + \partial_m\Psi\partial_n\Psi - \partial_m\Phi\partial_n\Psi) + 16\pi G\partial_m V\partial_n V \\ &- \Box(\partial_m\sigma\partial_n\sigma + \partial_m\partial^l E\partial_n\partial_l E) \end{aligned} \tag{5}$$

which is given in a representation of the gauge variables, under a generic **gauge transformation** of the form

$$\xi^i = (T, \partial^i L) \; and \; \xi_i = (T, \partial_i L) \tag{6}$$

with the definitions of the gauge invariant **Bardeen potentials** as:

$$\Phi = \phi + \frac{1}{3}\Delta E + \mathcal{H}\sigma$$

$$\Psi = \psi - \mathcal{H}\sigma - \dot{\sigma}$$

the **invariant velocity potential** as

$$V = v + E$$

and the **scalar shear potential**:

$$\sigma = B + \dot{E}$$

This can be easily confirmed if one assumes Longitudinal gauge, with $E = B = 0$ leading to $\Psi = \psi$ and $\Phi = \phi$.

Using all the above, we can see that we are led to express (4) as:

$$\Box h_{ij} = {M_{ij}}^{mn} S_{mn} = {M_{ij}}^{mn}(S^N_{mn} - \Box S^{\sigma}_{mn})$$

If we exploit the fact that the box operator is a linear operator and that it commutes with the TT projector, we end up with

$$\Box(h_{ij} + {M_{ij}}^{mn} S^{\sigma}_{mn}) = {M_{ij}}^{mn} S^N_{mn} \tag{7}$$

It can be proven that with the use of the gauge invariant potentials, S^N_{mn} is gauge invariant under the generic transformation (6) up to 2nd order.

If we define the part of the induced tensor perturbation as a variable $h^N_{mn} = h_{mn} + {M_{ij}}^{mn} S^{\sigma}_{mn}$, then we obtain

$$\Box h^N_{ij} = {M_{ij}}^{mn} S^N_{mn} \tag{8}$$

This is a wave equation with a source which is gauge invariant up to 2nd order. Since we are dealing with infinitesimal transformations and the geometry of our space-time manifold is to be considered the same, we can assume that the Lie derivative commutes with the D'Alabertian operator.

In a **matter dominated Universe** we have the expansion coefficient being $a_{(t)} \propto t^2$ leading to the Hubble parameter being $\mathcal{H} \propto \frac{2}{t}$ and so the wave equation of the induced tensor perturbation in Fourier space is:

$$h''_{ij} + \frac{4}{t} h'_{ij} + k^2 h_{ij} = S^{TT}_{ij}(\vec{k}, t) \tag{9}$$

(where for simplicity now h^N_{ij} is set as h_{ij}) Solving the homogeneous equations and forming the **Green function**, we can express the solution as:

$$h_{ij}(\vec{k}, t) = \int dt' G(\vec{k}, t, t') S^{TT}_{ij}(\vec{k}, t') \tag{10}$$

Results

The **Power spectrum** of the partial tensor mode is defined as:

$$\langle h_{ij}(\vec{k}', t) h_{ij}(\vec{k}, t)\rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P_h(\vec{k}, t)$$

which leads to

$$\langle h_{ij}(\vec{k}, t) h_{ij}(\vec{k}', t)\rangle = \int_0^t dt_1 dt_2 G(\vec{k}, t, t_1) G(\vec{k}', t, t_2) \langle S^{TT}_{ij}(\vec{k}, t_1) S^{TT}_{ij}(\vec{k}', t_2)\rangle$$

The expectation value of the quantity sourcing the part of the GW that we are interested in :

$$\langle S^{TT}_{ij}(\vec{k}, t_1) S^{TT}_{ij}(\vec{k}', t_2)\rangle \tag{11}$$

which is a **four point expectation value** at vectors (p, q, k, k').

Using ***Isserlis' Theorem*** or ***Wick's probabillity theorem*** we can reduce this expression to a combination of pairs and **directly connect the power spectrum** of the focused part of the **tensor modes** to the power spectrum of the primordial **scalar fluctuations** which induced the gravitational waves.

$$P_h \propto P^2_{\Psi}$$

Conclusions

The representation of gravitational waves in perturbation theory is sensitive to the gauge choice, making it crucial to identify gauge-invariant quantities as only these have a clear physical interpretation.

Using this approach, we can expand to the identification of gauge invariant parts of the GW to different Universes, with different symmetries (**Bianchi Universes**) or alternative **Theories of gravity**. Combining the study of the canonical structure and gauge transformations.

References

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